Estimator Eigenvalue Placement in Positive Real Control

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A numerical technique is presented to place the eigenvalues of the estimator in a positive real control environment. The estimator is based on a reduced-order model of the full system. The technique involves posing the problem as a constrained optimization problem. Two formulations are presented for solving the optimization problem that differ in their objective functions and in the way constraints are handled. Examples are given illustrating the pole placement for estimators based on an arbitrary second-order model, and on fourth- and sixth-order models of the DRAPER I tetrahedral truss structure.

Nomenclature

A,B = state and control penalty matrices used in index of performance
CC = control cost
F,G,H = state representation matrices for truth model
F_a = augmented state matrix involving truth model and estimator
F_{cl} = design closed-loop matrix for model $(F_m - G_m K)$
F_e = design closed-loop matrix for estimator $(F_m - K_F H_m)$
F_m, G_m, H_m = state representation matrices for reduced-order model
$G(s)$ = open-loop transfer matrix for truth model
$H_e(s)$ = open-loop transfer matrix for estimator
K = closed-loop feedback gain matrix
K_F = estimator gain matrix
M = matrix of eigenvalue sensitivities to changes in elements of Q
n = dimension of reduced-order model
p = dimension of input and output vectors
P = symmetric positive definite solution to Lyapunov equation
Q = arbitrary symmetric positive definite matrix used to find K_F
q = vector of independent elements of Q
S = arbitrary nonsingular matrix used to define Q
u_i = ith right eigenvector of F_e
v_i = ith left eigenvector of F_e
Δ_i = ith principal minor determinant of Q
Δ_{jk} = minor determinant for jkth element of S
$\bar{\lambda}$ = vector of desired eigenvectors for F_e

I. Introduction

THE use of feedback controllers that satisfy the positive real property are known to have robust stability properties and have been shown to be useful especially in regard to the control of flexible spacecraft.[1-3] In this case, the positive real controls can prevent destabilization of the closed-loop system due to control or observation spillover, and due to erroneous modal information. In a previous paper,[3] we applied a positive real synthesis technique to the design of large space structure controls. In this case, the controller eigenstructure was determined by ad hoc manipulations of an arbitrary symmetric positive definite matrix. To obtain controller eigenvalues with "sufficient" damping to give a reasonable closed-loop response was difficult in some cases when using a diagonal form for this matrix. A more general arbitrary matrix is desired.

In this paper, we discuss a technique for the positive real controller design that allows general eigenvalue placement, hence resulting in improved design flexibility. This technique is based on recasting the eigenvalue-placement problem as a constrained optimization problem. Two formulations for the solution of this problem are developed, and numerical results are presented for both algorithms.

II. Positive Real Systems

A time-invariant single-input single-output system is said to be positive real if the scalar system transfer function $G(s)$ satisfies the inequality $Re[G(s)] > 0$ for all $Re(s) > 0$, and the system is strictly positive real if $Re[G(s)] > 0$ for all $Re(s) \geq 0$. For a multivariable system, the Hermitian component of the frequency-response matrix must be positive semidefinite and positive definite, respectively. For a scalar system, positive realness implies the Nyquist plot lies strictly in the first and fourth quadrants. It is known that a positive real system is "dissipative," that is (generalized) energy in such a system is bounded above by the initial energy plus the control energy.[4] It is this dissipative nature of the positive real controller which makes it an attractive choice for the control of a conservative lightly damped structure.

The important stability theorem used to guarantee stability was first proposed by Popov.[5] The stability results can be stated as the following theorem.

Popov Stability Theorem: A system with forward transfer matrix $G(s)$ and feedback matrix $H(s)$ is guaranteed stable if one of the matrices is strictly positive real and the other is positive real.

For a flexible structure, the transfer matrix between ideal colocated actuators and velocity sensors is known to be positive real, hence a strictly positive real feedback will lead to a stable closed-loop response. Although realistic actuators and velocity sensors will *not* in general have a positive real transfer function matrix, Slater et al.[6] have shown how actuator dynamics can be viewed as a multiplicative perturbation to a baseline positive real model. These uncertainties can then be treated using the robustness results of Doyle and Stein,[7]

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Doyle,[8] and Safanov et al.[9] to derive sufficient conditions for guaranteed closed-loop stability. This stability will then hold even though the plant transfer function matrix will not be positive real over the entire frequency spectrum.

Control Design

The design of the positive real feedback controller is described in McLaren and Slater[3] and Slater.[10] Summarized briefly here, the controlled system is assumed to be in the form

$$\dot{x} = Fx + Gu \tag{1}$$

$$y = Hx \tag{2}$$

A positive real controller is designed using an observer model based on a reduced-order model of the controlled system. The reduced-order system is assumed to be in the form

$$\dot{x}_m = F_m x_m + G_m u \tag{3}$$

$$y = H_m x_m \tag{4}$$

where F_m, G_m, and H_m are the matrices of the reduced-order model, x_m is the n-dimensional reduced state associated with the reduced-order model, y is the p-dimensional output vector, and u is the p-dimensional input vector. The dynamics of the estimator are given by

$$\dot{\hat{x}} = F_m\hat{x} + G_m u + K_F(y - H_m\hat{x}) \tag{5}$$

while the feedback control function is assumed to be of the form

$$u = -K\hat{x} \tag{6}$$

The feedback gain matrix K can be chosen by any desired means. For our applications K is chosen to minimize an index of performance

$$\int_0^\infty (y^TAy + u^TBu)\,\mathrm{d}t \tag{7}$$

where the weighting matrices A and B are selected so as to place the closed-loop poles of the reduced model in Eqs. (3) and (4) at the acceptable locations (i.e., to place the poles of $[F_m - G_mK]$).

The observer transfer matrix relating the feedback variable $u' = -u$ to the output y can be found from Eq. (5) and (6) as

$$H_e(s) = K(sI - F_T)^{-1}K_F \tag{8}$$

with

$$F_T = (F_m - G_mK - K_FH_m) \tag{9}$$

The matrix K_F is an $n \times p$ matrix of estimator gains. $H_e(s)$ represents the compensation transfer matrix and must be chosen to yield closed-loop stability for the actual system given by Eqs. (1) and (2). $H_e(s)$ is guaranteed to be positive real if K_F is chosen to satisfy[10]

$$PF_T + F_T^TP = -Q \tag{10}$$

$$K_F = P^{-1}K^T \tag{11}$$

with both Q and P positive definite symmetric. Equation (10) may be rewritten as

$$PF_{cl} + F_{cl}^TP = (Q - K^TH_m - {}^T_mK) = -\hat{Q} \tag{12}$$

where

$$F_{cl} = (F_m - G_mK) \tag{13}$$

is now assumed to be known. Calculation of K_F given K and Q follows directly from Eqs. (11), (12), and (13).

The well-known properties of the Lyapunov equation (10) specify that with positive definite Q and P we are guaranteed that F_T is a stability matrix. In our solution technique using Eqs. (6) and (7), the choice of feedback gain matrix K leads to an asymptotically stable F_{cl}. The choice of either a positive definite Q, or a positive definite $\hat{Q}$ *does not* guarantee the positive real condition. $Q > 0$ does not guarantee that the solution P of the Lyapunov equation is positive definite since $\hat{Q}$ may not be positive definite. Similarly, although $\hat{Q} > 0$ is sufficient for $P > 0$, this does not automatically guarantee $Q > 0$ since the matrix $(K^TH_m + H_m^TK)$ will generally be indefinite. In a solution technique, the important necessary conditions to be enforced are $Q > 0$, $P > 0$. For a given choice of positive definite Q then, we need to be mindful that the definiteness of $\hat{Q}$ and of P must be ascertained separately. Clearly though, for "large" Q, the definiteness conditions on $\hat{Q}$ and P are easily met.

A choice of the matrix Q determines the estimator gain matrix K_F. Most of our previous studies have restricted this choice to a diagonal matrix, or at most one with ad hoc variations of a few matrix elements. It is clear, however, that this severely limits the eigenplacement of the controller and may lead to unsatisfactory control properties. Figure 1 shows the locus of eigenvalues of the nominal closed-loop estimator matrix $F_e = (F_m - K_FH_m)$ for a sixth-order system when Q is chosen as $Q = qI$, and illustrates the restrictions on eigenvalue placement when using a diagonal Q.

For very large values of q, the matrix $K_F \to 0$, and the closed-loop estimator matrix approaches the open-loop model F_m. As q decreases, the eigenvalues of the estimator move by various amounts. There is a lower limit on q at which P becomes indefinite. At this point, the elements of K_F become unbounded and one or more of the eigenvalues of F_e approach infinity. The lowest value on Fig. 1 ($q = 15.60$) is slightly above this limit, and for this case there is one very large negative eigenvalue, while the remaining eigenvalues remain finite.

A more general technique is required in order to determine a rational approach to the determination of the Q matrix, which can lead to more general eigenvalue placement and hence a satisfactory control design. In the present work, we

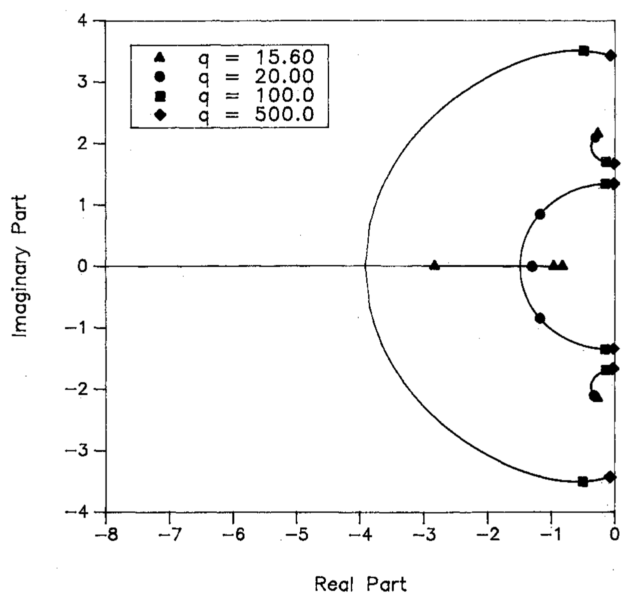

Fig. 1 Locus of eigenvalues of F_e for $Q = qI$ for sixth-order model.

give details of a numerical procedure to determine Q on the basis of eigenvalue placement. Prescribing a full Q matrix involves choosing the $n(n+1)/2$ independent elements, since Q is symmetric. If these values are to be chosen to fulfill an eigenvalue requirement, there are only n equations specifying these elements, so the problem is highly underdetermined.

There are no obvious correlations between the closed-loop controller eigenvalues of F_e and the choice of Q. In general, the K_F required to achieve a given set of desired eigenvalues is not unique, although the choice of Q to achieve a given K_F is also not unique. Finally it is easy to show that many eigenvalue placements are *not possible* within the positive real constraint.

III. Pole Placement for the Positive Real Estimator

The essence of this work is to show that the nonlinear pole-placement problem for the positive real estimator in Eqs. (10) and (11) can be converted to an equivalent optimization problem for which there are a variety of numerical algorithms. Two algorithms are presented that differ in their choice of performance index and in the method of handling constraints.

Eigenvalue Sensitivity Method

Consider the case where a nominal choice of Q in Eq. (12) results in a given set of eigenvalues for F_e, which are written in vector form as λ. These differ from the desired eigenvalues $\hat{\lambda}$ by an amount

$$\delta\lambda = \hat{\lambda} - \lambda \tag{14}$$

where $\dim(\lambda) = n$

Let the independent elements of Q be written in vector form as

$$q = [q_{11}\ q_{12}\ q_{13} \cdots]^T \tag{15}$$

where $\dim(q) = n(n+1)/2$

Small changes in q will cause changes in λ. To achieve the desired eigenvalue placement, we need to find the solution $\delta\lambda = f(q)$, which may be an underdetermined set of nonlinear equations. If we assume the changes in λ and q are small, then we can use linear theory and write

$$\delta\lambda = M\delta q \tag{16}$$

where M is a matrix of partial derivatives. M is of dimension $n \times n(n+1)/2$, hence there are a multiplicity of solutions for δq. To stay within the linear regime, one approach is to find the "smallest" δq to minimize the change in Q, i.e., the least-squares solution

$$\delta q = M^{\dagger}\delta\lambda \tag{17}$$

where

$$M^{\dagger} = M^T(MM^T)^{-1} \tag{18}$$

If the original $\delta\lambda$ is large, then the δq given by Eq. (17) is large and the linear theory is no longer valid. Hence, we must break the actual $\delta\lambda$ into a number of smaller steps for which the linear theory is applicable, and proceed sequentially to the desired set of eigenvalues. Two problems must be solved to apply this technique. The first is to compute M, the matrix of partials, and the second is to impose the positive definiteness constraints on P and Q to ensure positive realness of the resulting transfer matrix.

Eigenvalue Sensitivities

The M matrix is determined by evaluating the sensitivities of the estimator eigenvalues to parameter changes. These sensitivities are found from the properties of the eigenvalues and eigenvectors of F_e. Let λ_i be the ith eigenvalue of F_e. Then the right eigenvector u_i and the left eigenvector v_i satisfy the relations

$$F_e u_i = \lambda_i u_i \tag{19}$$

$$v_i^* F_e = \lambda_i v_i^* \tag{20}$$

where $()^*$ indicates conjugate transpose and u_i, v_i are chosen to satisfy the orthogonality condition

$$v_i^* u_j = \delta_{ij} \tag{21}$$

Differentiating Eq. (19) with respect to the jkth element of F_e, premultiplying by v_i^*, and using Eq. (20) and the orthogonality condition of Eq. (21), we can obtain the sensitivity relationship

$$\frac{\partial\lambda_i}{\partial F_{ejk}} = v_i^*\left(\frac{\partial F_e}{\partial F_{ejk}}\right)u_i \tag{22}$$

Using the property that

$$\left(\frac{\partial F_e}{\partial F_{ejk}}\right)_{im} = \delta_{ij}\delta_{km} \tag{23}$$

we find that

$$\left(\frac{\partial\lambda_i}{\partial F_{ejk}}\right) = v_{ij}^* u_{ik} \tag{24}$$

where v_{ij}^* and u_{ik} are the jth element of v_i^* and the kth element of u_i, respectively.

Using these results, we can proceed to find the relationship of the eigenvalue λ_i to a change in the jkth element of Q through a succession of chain-rule operations involving K_F [Eq. (11)], P [Eq. (12)], and finally Q. Performing these manipulations, we have finally

$$S_{jk}^i = \frac{\partial\lambda_i}{\partial\delta q_{jk}} = v_i^* P^{-1}\left(\frac{\partial P}{\partial\delta q_{jk}}\right)K_F H_m u_i \tag{25}$$

where S_{jk}^i is the partial derivative of λ_i with respect to a change in the jkth element of Q, and

$$\left(\frac{\partial P}{\partial\delta q_{jk}}\right)$$

is a matrix obtained by differentiating Eq. (12). This gives the Lyapunov equation

$$\left(\frac{\partial P}{\partial\delta q_{jk}}\right)F_{cl} + F_{cl}^T\left(\frac{\partial P}{\partial\delta q_{jk}}\right) = -\left(\frac{\partial Q}{\partial\delta q_{jk}}\right) \tag{26}$$

where because of the required symmetry of Q

$$\left(\frac{\partial Q}{\partial\delta q_{jk}}\right)_{rs} = (\delta_{rj}\delta_{sk}) + (\delta_{rk}\delta_{sj}) \tag{27}$$

The matrix M in Eq. (16) is formed by the appropriate combination of the sensitivity elements S_{jk}^i. Evaluation of the elements of M as above requires the solution of the $n(n+1)/2$ Lyapunov equations contained in Eq. (26) to obtain the sensitivities

$$\left(\frac{\partial P}{\partial\delta q_{jk}}\right)$$

Each solution is related, however, since they differ only on the right-hand side of Eq. (26). Since most techniques for the solution of low-order Lyapunov equations require an LU decomposition, the entire set of matrix sensitivity solutions reduce to one LU decomposition and $n(n+1)/2$ elimination solutions to a set of linear equations, thereby making the numerical problem quite tractable.

Generally, the systems we deal with will have complex eigenvalues, hence the M as defined here is complex. To remain in the real domain for ease of computation, it is convenient to redefine M and $\delta\lambda$ by partitioning Eq. (16) into real and imaginary components

$$M = \begin{bmatrix} M_r \\ M_i \end{bmatrix}, \qquad \delta\lambda = \begin{bmatrix} \delta\lambda_r \\ \delta\lambda_i \end{bmatrix} \tag{28}$$

with the subscripts r and i indicate the real and imaginary parts of the respective quantities. For complex conjugate pairs of poles only one root is used, thus not changing the overall order of the problem, and ensuring that M is not rank deficient.

Also, if it is not necessary to place all n eigenvalues of a problem, then only those eigenvalues which are explicitly required to be placed need be involved in the analysis. Similarly, if the imaginary part of the estimator eigenvalues are not required to be placed, then one can simply delete the parts of Eq. (28) that deal with the imaginary part and then solve the problem as before.

Finally, in the numerical implementation of this solution, care must be taken to ensure that the change vector $\delta\lambda$ requested is "small," so that one remains within the linear region (where changes in Q cause a change in λ that is close to the change predicted by the linear analysis). Hence, in the examples shown here we solve the linear problem repeatedly for a number of steps of the form $\alpha\delta\lambda$, where the α is chosen to make the $\alpha\delta\lambda$ small.

The Positivity Constraints

If, during the eigenvalue placement as already outlined, the Q matrix crosses a positive definite boundary and becomes and remains indefinite, the final $H_e(s)$ will violate the Popov stability theorem as stated earlier for a positive real plant $G(s)$ (even though the matrix P may indeed still be positive definite!). Therefore, at a positive definite boundary of Q, we must alter the change vector δq in such a way that the new Q will not be indefinite, yet still move the estimator eigenvalues in the desired direction.

The constraints on the Q matrix being positive definite correspond to constraints on the principal minor determinants of Q,

$$\Delta_i > 0 \qquad \text{for } i = 1,2,\ldots,n \tag{29}$$

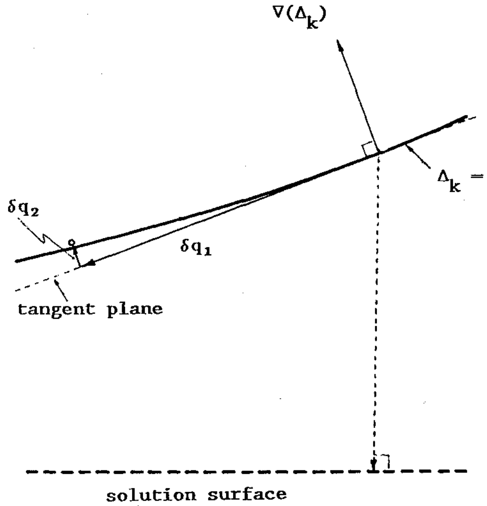

Fig. 2 Representation of solution at a constraint surface.

where Δ_i is the ith principal minor determinant of Q. The positive definite boundaries are the surfaces $\Delta_i = 0$, which are nonlinear constraint surfaces in the $R^{n(n+1)/2}$ space.

Consider a change vector δq that causes Q to cross a positive definite boundary, and let the first constraint that is violated be constraint k, i.e., $\Delta_k = 0$ at the point. The normal to the tangent plane of the constraint surface at the point of crossing is found as $\nabla(\Delta_k)$. This gradient vector is always in a direction of increasing Δ_k, that is, back into the positive definite region of space. The elements of the gradient vector are the matrix minors of Q for the appropriate element, and can be easily evaluated numerically at the point of interest by successive determinant evaluations, or for low-order problems by analytical expressions.

Consider that we are already at a positive definite boundary $\Delta_k = 0$. Any further moves to accomplish eigenplacement should be done along this boundary. As a linear approximation to the boundary, we move in the tangent plane, which is completely determined by the gradient vector at the point of contact with the surface. To remain within our minimum norm approach, we simply append another constraint to our constraint equations (16), which then become

$$M_1\delta q_1 = \begin{bmatrix} M \\ \nabla(\Delta_k)I \end{bmatrix} \qquad \delta q_i = \begin{bmatrix} \delta\lambda \\ 0 \end{bmatrix} = \delta\lambda' \tag{30}$$

and the solution will be

$$\delta q_1 = M_1^\dagger \delta\lambda' \tag{31}$$

The new position may still not be in the positive definite region due to the convexity of the constraint surface at the point of contact, so one last move must be made back into the positive definite region. The most direct method would be to simply move in the direction of the gradient vector at our new point until we cross the constraint surface. However, this can lead to problems if this move is done without reference to any eigenplacement or minimum norm requirements. In practice, we find that in some regions the constraint gradient is very nearly aligned with the direction of greatest change of λ. Thus, even small changes back to the constraint surface sometimes produce objectionable change in system eigenvalues.

In parallel with our algorithm so far, we simply append a constraint to Eq. (16) as before, but now we wish no movement in the eigenvalues while increasing the principal minor in question. Suppose the principal minor determinant at our new point is still some negative quantity $-\beta^2$. Then the problem becomes

$$M_1\delta q_2 = \begin{bmatrix} 0 \\ \beta^2 \end{bmatrix} \tag{32}$$

which can be solved for the second vector δq_2. Finally then, the new change vector becomes

$$\delta q = \delta q_1 + \delta q_2 \tag{33}$$

A two-dimensional representation of what is occurring at a constraint boundary is given in Fig. 2.

Although it is necessary and sufficient that *both* Q and P be positive definite for $H_e(s)$ to be strictly positive real, in practice one need only consider the constraint on Q because of the way the filter matrix K_F is evaluated. The initial Q is chosen such that $P>0$. If P is to become indefinite, then some small step must cause P to go through the singular point $\det(P) = 0$. However, as P comes closer to the singular point, P^{-1} will become very large, hence K_F will become very large due to Eq. (11). This will tend to significantly change the eigenvalues of F_e, and the constraint that the eigenvalue placement be satisfied for that particular iteration tends to alter Q such that P^{-1} decreases, thus moving P further from being singular.

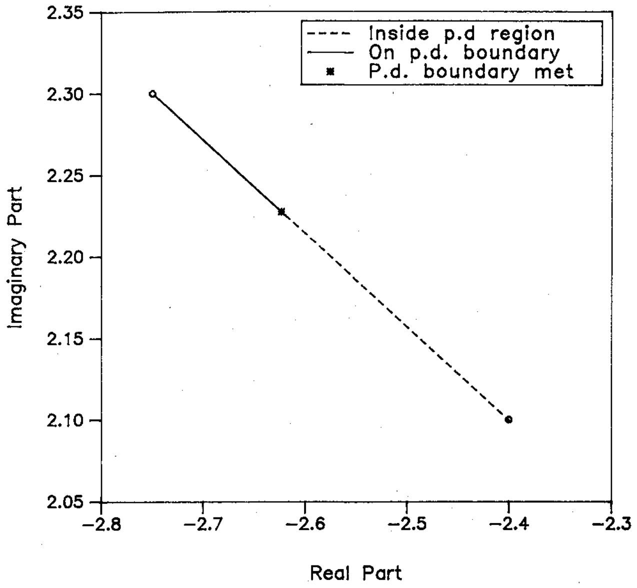

Fig. 3 Convergence history of estimator eigenvalues for second-order model.

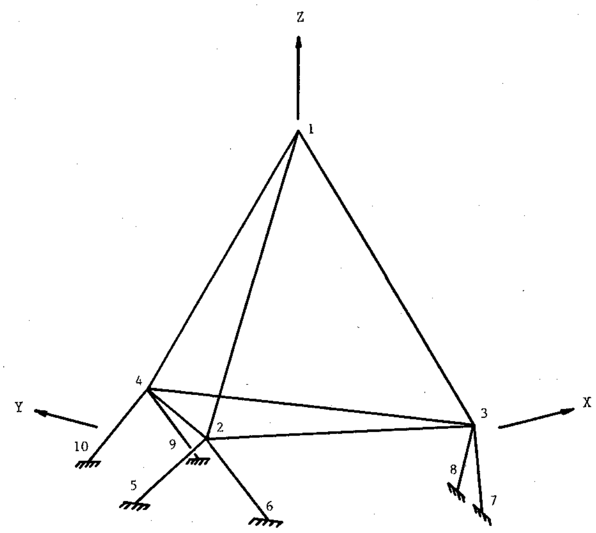

Fig. 4 The DRAPER I tetrahedral truss structure.

Example 1

We illustrate the procedure outlined above with a simple second-order example. Consider the system whose reduced-order model has been chosen as

$$F_m = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad G_m = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H_m = [0 \quad 1]$$

A feedback matrix has been chosen as $K = [0 \quad 2]$, which gives the eigenvalues of $(F_m - G_m K)$ as $-1.0 \pm j\,1.0$.

The estimator gain matrix was chosen initially to fulfill the strict positive real requirements of the theory from a nominal choice of a symmetric Q as

$$Q = \begin{bmatrix} 7 & -3 \\ & 7 \end{bmatrix}$$

where only the upper triangular part of the symmetric Q is shown. This Q places the eigenvalues of $F_e = (F_m - K_F H_m)$ at $-0.7628 \pm \mathrm{j}\,1.4021$. From here, the eigenvalue-placement algorithm was applied to move the eigenvalues of F_e to $-2.4 \pm j\,2.1$ (at this stage an arbitrary eigenplacement). The total $\delta\lambda$ desired is then $1.63272 \pm j\,0.6979$, which (it is easily checked) is well outside the range where the linear analysis used is applicable. The step size is an upper bound to the allowable desired change vector $\delta\lambda$ at any step, and was set to 0.002 for this case. This step size is small enough to ensure that we are within the linear region of our algorithm, and can be estimated by comparing the desired and actual changes for one step.

After a number of small steps, the desired poles are reached, at which point

$$Q = \begin{bmatrix} 6.8095 & 5.1317 \\ & 8.0533 \end{bmatrix}$$

During the steps to this set of eigenvalues, no positive definite constraint boundary was encountered. From this position, a new set of desired poles was specified as $-2.75 \pm j\,2.3$, and the eigenvalue-placement algorithm applied again. Figure 3 shows the resulting movement of the eigenvalues. A positivity boundary is met during the placement, but the algorithm compensates by appending the extra constraints to the sensitivity matrix, as in Eqs. (30) and (32).

At the desired eigenvalues, the final Q was

$$Q = \begin{bmatrix} 4.1335 & 3.7837 \\ & 7.4637 \end{bmatrix}$$

Note that for this second-order problem, once the extra constraint is added, the modified M becomes square, and at each step the change vectors δq_1 and δq_2 become uniquely determined.

Example 2

The second example presented here is for control of the DRAPER I tetrahedral truss structure,[11] shown in Fig. 4, which has been used by various authors as a generic spacecraft model. The performance is measured as the ability to control the x and y positions (line-of-sight error) of the top vertex to zero. The right-angled bipods that connect the truss to the ground take on the dual purposes of force actuators and velocity sensors, hence the colocation property is satisfied and the model transfer function will be positive real.

A fourth-order model of the structure is formed using the two modes of lowest frequency, which have frequencies $\omega_1 = 1.342$ rad/s and $\omega_2 = 1.665$ rad/s, respectively. Our system matrices therefore become, in modal form

$$F_m = \begin{bmatrix} [0] & [I] \\ -[\omega^2] & [0] \end{bmatrix}$$

where

$$[\omega^2] \quad \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

There are six actuator/sensor pairs, with $G_m = H_M^T$. The G_m matrix is evaluated from the known modal matrix and structure geometry (see, for example, Ref. 5).

The feedback gain matrix K was chosen to minimize the index of performance given in Eq. (7) with $A = \mathrm{diag}\,\{10^3\}$ and $B = \mathrm{I}$. The resulting poles of $F_{cl} = (F_m - G_m K)$ are at -1.3177, -1.4931, -2.7169, and -4.5072. For the estimator, a set of desired closed-loop pole locations was specified as $-4.0 \pm j\,2.5$ and $-4.5 \pm j\,3.0$. The estimator gain matrix is chosen initially using the positive real equations from a nominal diagonal $Q = \text{diag}\{30.0\}$, which places the eigenvalues of F_e at $-0.6555 \pm j\,1.2321$ and $-1.5480 \pm j\,0.8986$. The eigenvalue-placement algorithm was applied using a step size of 0.001. The eigenvalue paths are shown in Fig. 5, and again we see that we meet a positive definite constraint during the placement.

At the desired eigenvalue locations, the Q matrix is

$$Q = \begin{bmatrix} 23.929 & 6.1181 & 29.916 & 6.3219 \\ & 24.373 & 3.7502 & 31.057 \\ & & 38.780 & 4.0889 \\ & & & 41.758 \end{bmatrix}$$

It is important to note that this *not* a unique Q to achieve the desired eigenplacement. Starting from a different location in the complex s plane, or even using a different step size, may give a very different final Q.

General Non-Linear Optimization Method

For some problems, the eigenvalue sensitivity method just described was found to exhibit several drawbacks. First, in some cases the steps suggested to choose δq will not work. This can occur, for example, when the desired eigenvalue locations lie outside the region where positive real control is possible. Second, in some cases investigated, the surfaces of constant eigenvalue and surfaces of constant principal minor determinant are almost parallel at some points in the iteration. In this case, $\delta\lambda$ must be chosen very small or else the δq_2 calculated may be very large. This problem may sometimes be circumvented by starting from a different location in the s plane, but there is no general theory available to choose alternate starting points. Third, the linear solution may also exhibit very slow convergence in regions where the constraint surface has high curvature. These regions are not known a priori because of the high dimensionality and complexity of the constraint surfaces. In such a case, step sizes may need to be very small in order that the linear approximation be valid. For larger step sizes, the nonlinear terms in the expansion for the constraints become significant.

To resolve some of these difficulties, an alternate form of the pole-placement optimization algorithm is proposed. Rather than seek a minimum norm on δq for a specified eigenvalue change, we now seek only to minimize the norm of the error in eigenvalue placement, i.e.,

$$\min \sum_{i=1}^{n} |\lambda_i - \hat{\lambda}_i|^2 \tag{34}$$

For most problems this minimum is zero, and may be achieved for a wide range of possible Q matrices. To further simplify our search for a positive definite symmetric Q, we search rather for a factored $(n \times n)$ matrix s, where

$$Q = S^T S \tag{35}$$

If desired, a special form of S (e.g., a triangular S) instead of a full S may be used to reduce the number of variables. The definiteness constraints are eliminated and replaced by the sole constraint

$$\det(S) \neq 0 \tag{36}$$

which in fact will usually be a nonactive constraint. It will only become active during the solution procedure if the set of desired poles lies outside the positive real region, or if a traverse across the positive real boundary is requested.

As in the previous method, only those eigenvalues required to be placed need to included in the sum in Eq. (34), and if the imaginary parts of the eigenvalues are free, then these components can be deleted from the objective. This objective has the advantage over that used previously in that if the set of desired eigenvalues is not a subset of the achievable set, then the minimization remains well-posed and will converge to a set of poles as close as possible to the desired set. In principle, this technique could be used to evaluate numerically the positive real region for particular pole groupings.

Derivatives of the performance index (34) are no more difficult to evaluate than the eigenvalue sensitivities evaluated previously. Using the same methodology, the first derivative of the objective function (34) with respect to the elements of S is

$$\frac{\partial J}{\partial S_{jk}} = \sum_{i=1}^{n} 2|\lambda_i - \hat{\lambda}_i| \left(\frac{\partial \lambda_i}{\partial S_{jk}}\right) \tag{37}$$

where, in parallel with the first algorithm,

$$\frac{\partial \lambda_i}{\partial S_{jk}} = v_i^* P^{-1} \left(\frac{\partial P}{\partial S_{jk}}\right) K_F H_m u_i \tag{38}$$

P is the positive definite solution to

$$PF_{cl} + F_{cl}^T P = -(S^T S - K^T H_m - H_m^T K) \tag{39}$$

and

$$\left(\frac{\partial P}{\partial S_{jk}}\right)$$

is a matrix obtained by differentiating Eq. (39), which gives the Lyapunov equation

$$\left(\frac{\partial P}{\partial S_{jk}}\right) F_{cl} + F_{cl}^T \left(\frac{\partial P}{\partial S_{jk}}\right) = -\left[\left(\frac{\partial S}{\partial S_{jk}}\right)^T S + S^T \left(\frac{\partial S}{\partial S_{jk}}\right)\right] \tag{40}$$

The solution to the preceding constrained optimization problem was found using the subroutine package GRG2 (Generalized Reduced Gradient) developed by Lasdon and Waren[12] in 1982.

Examples 1 and 2 Revisited

When examples 1 and 2 are solved again using this second formulation, the poles are again as desired, and the final Q

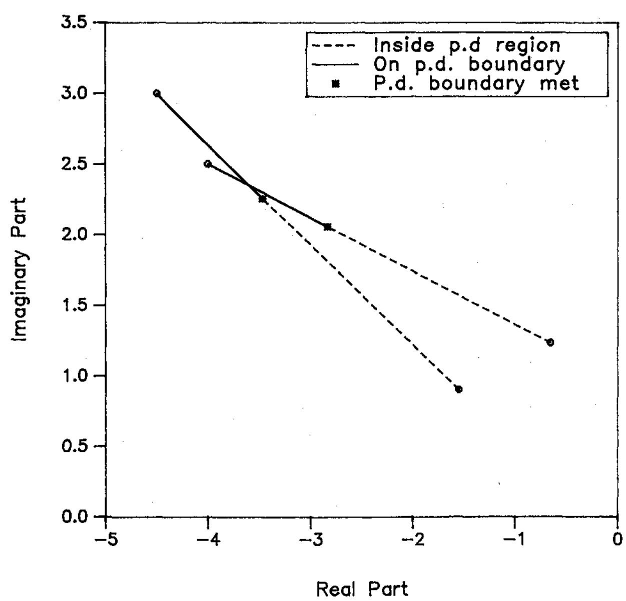

Fig. 5 Convergence history for estimator eigenvalues for fourth-order model of the DRAPER I structure.

matrices obtained are as follows:

$$Q = \begin{bmatrix} 5.5826 & 4.8579 \\ & 8.1710 \end{bmatrix}$$

$$Q = \begin{bmatrix} 24.131 & -3.9992 & 31.307 & -0.7582 \\ & 83.110 & -8.7024 & 90.621 \\ & & 41.045 & -5.1034 \\ & & & 100.69 \end{bmatrix}$$

Note that these Q matrices are very different from those evaluated using the first algorithm, yet accomplish the same pole placement. This underscores the nonuniqueness of the solution matrix Q, as well as the nonuniqueness of K_F (in general). Since for this case we used the GRG2 code, no external search patch, such as that in Figs. 3 and 5, was found.

For the two methods presented, neither appears to produce a consistent control-system-performance superiority, although numerically the latter approach is clearly more efficient and robust.

Example 3

The last example is a sixth-order model of the DRAPER I truss structure. This example was used to generate the eigenvalue locus given in Fig. 1, which illustrates the estimator eigenvalue placement when using a diagonal $Q = qI$. In this example, we illustrate the additional flexibility in eigenvalue placement afforded by allowing an arbitrary (nondiagonal) $Q = S^TS$. The modes kept in the model were modes 1, 2, and 5 (in Ref. 3 these modes were determined as "most" important). Mode 5 has frequency $\omega_5 = 3.398$ rad/s. The feedback gain matrix K is chosen to minimize Eq. (7) with $A = \text{diag}\{10^3\}$ and $B = I$. The resulting poles of F_{cl} are at -1.2754, -1.4931, -2.7169, -5.8560, and $-0.6509 \pm j\, 3.0028$. The estimator gain matrix is chosen from an initial choice of $S = S_0 = \text{diag}\{\sqrt{50}\}$, which places the eigenvalues of F_e at $-0.347 \pm j\, 1.322$, $-0.296 \pm j\, 1.781$, and $-1.289 \pm j\, 3.396$.

A set of desired pole locations is specified as $-1.70 \pm j\, 1.70$, $-1.80 \pm j\, 1.80$, and $-3.00 \pm j\, 3.50$. This set of desired poles is chosen rather arbitrarily—the point of the exercise being to show the efficiency of the eigenvalue-placement algorithm. In previous work,[3] we had found that the inability to do this with a diagonal Q (as shown in Fig. 1) led to poor controller performance.

Using the second formulation, this problem was solved using the optimization package GRG2, and at the desired eigenvalue locations, the Q matrix is

$$Q = \begin{bmatrix} 64.76 & -15.00 & -19.90 & -4.272 & 29.70 & -21.36 \\ & 74.06 & -11.86 & 27.70 & 33.84 & -0.953 \\ & & 109.3 & 3.445 & -66.26 & 52.38 \\ & & & 22.57 & 9.321 & -2.100 \\ & & & & 60.10 & -35.29 \\ & & & & & 30.38 \end{bmatrix}$$

Figure 6 shows the eigenplacement with respect to the initial eigenvalues of the estimator. It is obvious that a significant increase in the damping ratios for the eigenvalues of F_e has been achieved while leaving the damped frequencies almost constant. This solution required approximately 60 s CPU time on an Amdahl 470 computer.

The question remains whether or not this eigenvalue placement results in improved system performance over a diagonal Q. To address this question, we form a positive real controller for this system, using a diagonal Q of the form $Q = qI$, which has the same "control cost" as the controller using the general pole placement presented here. The control cost here is defined as

$$CC = \int_0^\infty (u^TBu)\, \mathrm{d}t = x_{a0}^T Z x_{a0} \tag{41}$$

where $x_a^T = \{x^T\ \hat{x}^T\}$, Z satisfies

$$ZF_a + F_a^TZ = -K_a \tag{42}$$

and

$$F_a = \begin{bmatrix} F & -GK \\ K_FH & F_T \end{bmatrix} \tag{43}$$

$$K_a = \begin{bmatrix} 0 & 0 \\ 0 & K^TRK \end{bmatrix} \tag{44}$$

The cost is evaluated for a particular set of initial conditions x_{a0}. In this case, x_{a0} was chosen as a unit displacement in the $x-y$ plane of the top vertex of the truss, equally shared in the x and y directions.

If there is no model error, i.e., if F, G, and H are F_m, G_m, and H_m, respectively, then the predicted eigenvalues of F_{cl} and F_e are indeed the true system closed-loop eigenvalues. For this case, shown in Fig. 7, the system whose poles have been placed to improve their damping exhibits much better error response than the system whose poles were chosen on the basis of a diagonal Q, even though the same control energy is used.

If, however, we use the full system matrices for F, G, and H, we introduce additional unmodeled modes, as well as control and observation spillover. For this case, we observe the time response as shown in Fig. 8. The significant increase in the damping ratios of the estimator eigenvalues does not translate, in this case at least, to a significant improvement in the line of sight (LOS) error response. For this problem, the residual modes in the system (those modes not in the controller) are excited by this disturbance, and the improved damping in the estimator modes does not affect them. This is a manifestation of the general problem of modal approximation and model-order reduction. When estimator feedback is used as the basis for controller design, the predicted closed-loop poles are simply the poles of F_{cl} and F_e. In practice, when a reduced-order model is used to design the estimator as in our examples, the

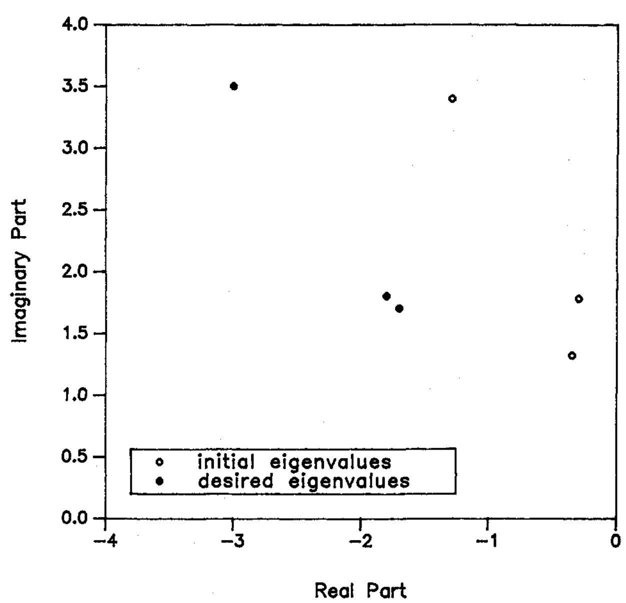

Fig. 6 Initial and final eigenvalue placement for sixth-order model of the DRAPER I structure.

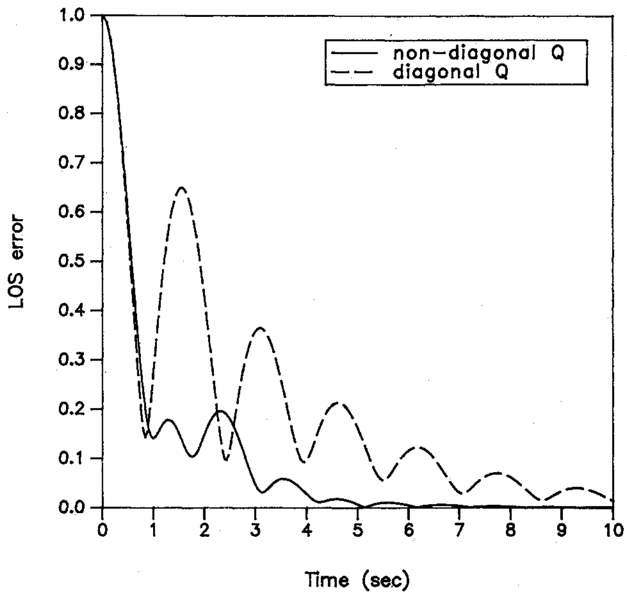

Fig. 7 LOS error response using a reduced-order system approximation.

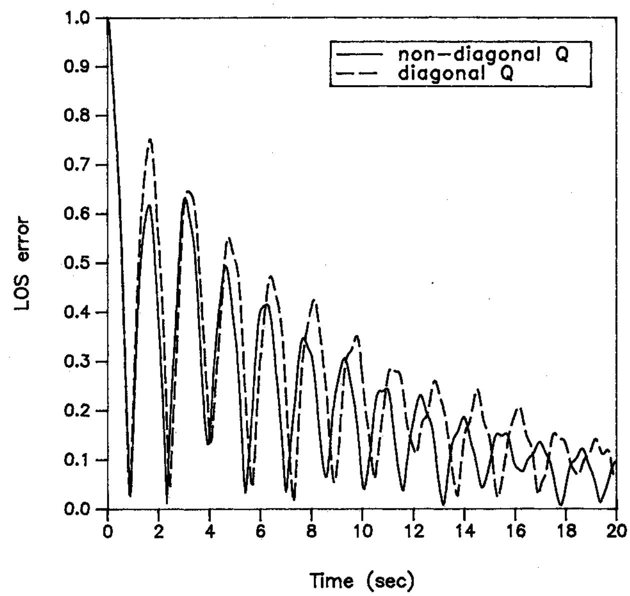

Fig. 8 LOS error response of full system.

actual closed-loop poles may be quite unpredictable, depending on the fidelity of the approximate model.

IV. Conclusions

A new technique has been presented for the determination of positive real transfer matrices with desired eigenvalues. The technique casts the problem as a constrained optimization problem that is solved iteratively using gradient methods. Two solution algorithms have been presented which differ in the optimization problem formulation. The second algorithm, which is based on nonlinear optimization, was found to be superior for the examples used in terms of computer time, overall convergence characteristics, and robustness. The new algorithms were successfully applied to a simple second-order problem, and to fourth- and sixth-order models of the DRAPER I structure. Significant changes in the estimator eigenvalue positions have been achieved while retaining the positive realness of the transfer matrix. This technique promises to give greater flexibility in design when using positive real control.

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